Detection with Vector Observations

- As before, there are two hypotheses H_0 and H_1.
- The measurement (or observation) consists of n random variables $Y_1, \dots, Y_n$ that we organize into a random vector $\underline{Y}$.

$$\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

Discrete Case

$P_{\underline{Y}|H_0}(\underline{y})$ if H_0 occurs

$P_{\underline{Y}|H_1}(\underline{y})$ if H_1 occurs

Continuous Case

$f_{\underline{Y}|H_0}(\underline{y})$ if H_0 occurs

$f_{\underline{Y}|H_1}(\underline{y})$ if H_1 occurs

- The detector (or decision rule) $D(\underline{y})$ outputs 0 if it decides H_0 occurred and 1 if it decides H_1 occurred, using only its input $\underline{y}$.
  → Partitions the range of $\underline{Y}$ into decision regions
  $A_0 = \{\underline{y} \in R_{\underline{Y}} : D(\underline{y}) = 0\}$, $A_1 = \{\underline{y} \in R_{\underline{Y}} : D(\underline{y}) = 1\}$.
- Probability of Error: $P_e = P_{FA}\, \mathbb{P}[H_0] + P_{MD}\, \mathbb{P}[H_1]$

$P_{FA} = \mathbb{P}[\underline{Y} \in A_1 \mid H_0]$ $\quad$ $P_{MD} = \mathbb{P}[\underline{Y} \in A_0 \mid H_1]$

False Alarm $\quad$ Missed Detection

- Likelihood Ratio: $L(\underline{y}) = \begin{cases} \dfrac{P_{\underline{Y}|H_1}(\underline{y})}{P_{\underline{Y}|H_0}(\underline{y})} & \underline{Y} \text{ is discrete} \\ \dfrac{f_{\underline{Y}|H_1}(\underline{y})}{f_{\underline{Y}|H_0}(\underline{y})} & \underline{Y} \text{ is continuous} \end{cases}$

- <u>Maximum Likelihood (ML) Decision Rule:</u>

$$\boxed{D^{ML}(\underline{y}) = \begin{cases} 1 & L(\underline{y}) \geq 1 \\ 0 & L(\underline{y}) < 1 \end{cases} = \begin{cases} 1 & \ln(L(\underline{y})) \geq 0 \\ 0 & \ln(L(\underline{y})) < 0 \end{cases}}$$

$\ln(L(\underline{y}))$ is the log-likelihood ratio

- <u>Maximum a Posteriori (MAP) Decision Rule:</u>

$$\boxed{D^{MAP}(\underline{y}) = \begin{cases} 1 & L(\underline{y}) \geq \dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} \\ 0 & L(\underline{y}) < \dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]} \end{cases} = \begin{cases} 1 & \ln(L(\underline{y})) \geq \ln\left(\dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \\ 0 & \ln(L(\underline{y})) < \ln\left(\dfrac{\mathbb{P}[H_0]}{\mathbb{P}[H_1]}\right) \end{cases}}$$

→ MAP rule is optimal: it minimizes the probability of error. Equivalent to ML rule only when $\mathbb{P}[H_0] = \mathbb{P}[H_1] = \frac{1}{2}$.

- **Example:** Given that H_0 occurs, $\underline{Y}$ is Gaussian$(\underline{\mu}_0, \Sigma_{\underline{Y}})$.
Given that H_1 occurs, $\underline{Y}$ is Gaussian$(\underline{\mu}_1, \Sigma_{\underline{Y}})$.

$$\underline{\mu}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \underline{\mu}_1 = \begin{bmatrix} +1 \\ +1 \end{bmatrix} \qquad \Sigma_{\underline{Y}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$\mathbb{P}[H_0] = \mathbb{P}[H_1] = \frac{1}{2}$.

Determine the optimal decision rule.

→ Since $\mathbb{P}[H_0] = \mathbb{P}[H_1] = \frac{1}{2}$, the optimal MAP rule is the same as the ML rule. First, we need the likelihood ratio.

$$f_{\underline{Y}|H_0}(\underline{y}) = \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma_{\underline{Y}})}} \exp\left(-\frac{1}{2}(\underline{y} - \underline{\mu}_0)^T \Sigma_{\underline{Y}}^{-1} (\underline{y} - \underline{\mu}_0)\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\begin{bmatrix} y_1 - (-1) & y_2 - (-1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 - (-1) \\ y_2 - (-1) \end{bmatrix}\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left((y_1 + 1)^2 + (y_2 + 1)^2\right)\right)$$

$$f_{\underline{Y}|H_1}(\underline{y}) = \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma_{\underline{Y}})}} \exp\left(-\frac{1}{2}(\underline{y} - \underline{\mu}_1)^T \Sigma_{\underline{Y}}^{-1} (\underline{y} - \underline{\mu}_1)\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\begin{bmatrix} y_1 - (+1) & y_2 - (+1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 - (+1) \\ y_2 - (+1) \end{bmatrix}\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left((y_1 - 1)^2 + (y_2 - 1)^2\right)\right)$$

$$f_{\underline{Y}|H_0}(\underline{y}) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\left((y_1+1)^2+(y_2+1)^2\right)\right)$$

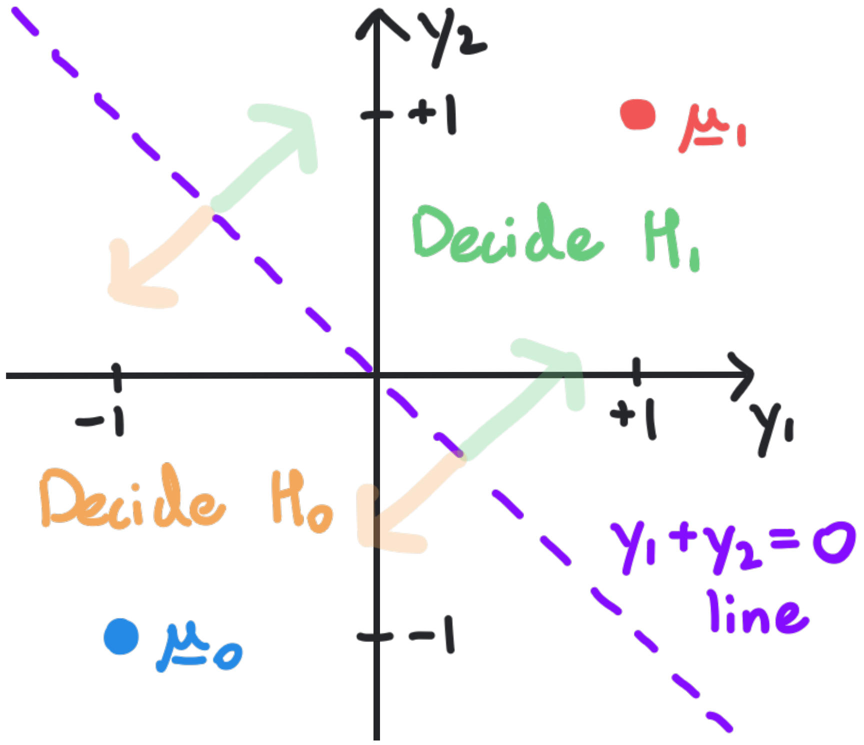
$$f_{\underline{Y}|H_1}(\underline{y}) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\left((y_1-1)^2+(y_2-1)^2\right)\right)$$

Likelihood Ratio

$$L(\underline{y}) = \frac{f_{\underline{Y}|H_1}(\underline{y})}{f_{\underline{Y}|H_0}(\underline{y})} = \frac{\cancel{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}\left(\cancel{y_1^2}+2y_1\cancel{+1}+\cancel{y_2^2}+2y_2\cancel{+1}\right)\right)}{\cancel{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}\left(\cancel{y_1^2}-2y_1\cancel{+1}+\cancel{y_2^2}-2y_2\cancel{+1}\right)\right)} = \exp\left(2(y_1+y_2)\right)$$

ML Rule

$$D^{ML}(\underline{y}) = \begin{cases} 1 & L(\underline{y}) \geq 1 \\ 0 & L(\underline{y}) < 1 \end{cases} = \begin{cases} 1 & \exp(2(y_1+y_2)) \geq 1 \\ 0 & \exp(2(y_1+y_2)) < 1 \end{cases} = \begin{cases} 1 & y_1+y_2 \geq 0 \\ 0 & y_1+y_2 < 0 \end{cases}$$



Intuitively, the ML rule selects the hypothesis whose mean is closest to the observed vector.

Probability of error can be determined using the fact that $Y_1 + Y_2$ is Gaussian under H_0 and under H_1 since it is a linear function.